Every number is the beginning of a power of 2

I do not know when you were born, but I am sure that your birthyear is the beginning of a power of 2. I do not know exactly how many grains of sand there are in the sea, but this number is surely the beginning of a power of 2. *Given any natural number, I know that this number is the beginning of a power of* 2 (and in fact it is the beginning of infinitely many powers of 2).
For example, consider the number 123. The power 2^{90} starts with the digits 123:

$$2^{90} = 1237940039285380274899124224\,.$$

You may check with a computer that the powers 2^{379}, 2^{575}, 2^{864} also start with the digits 123, and I claim that there are infinitely many powers of 2 with this property!

Given any natural number A, we prove that there is a power of 2 starting with the digits of A (as a small challenge adapt the proof and see that there are infinitely many powers of 2 with this property). We have to find some power 2^n such that for some integer number $k \geqslant 0$ we have

$$A \cdot 10^k \leqslant 2^n < (A+1)10^k\,.$$

Indeed, this ensures that the first digits of 2^n are those of A, and then there are k further digits which can be arbitrary. This condition can be rewritten using decimal logarithms:

$$\log(A) + k \leqslant n\log(2) < \log(A+1) + k\,.$$

Now plug in the floor function[1] and the fractional part[2] of the above numbers:

$$\lfloor \log(A) \rfloor + \{\log(A)\} + k \leqslant \lfloor n\log(2) \rfloor + \{n\log(2)\} < \lfloor \log(A+1) \rfloor + \{\log(A+1)\} + k\,.$$

I leave you to deal with the easy case where $A+1$ is a power of 10, so we can assume that $\log(A)$ and $\log(A+1)$ have the same floor function. Moreover, let's choose

$$k = \lfloor n\log(2) \rfloor - \lfloor \log(A) \rfloor$$

(notice that, provided that n is sufficiently large, k will be a positive integer). The inequalities then simplify a lot: to solve our problem it then suffices to find some sufficiently large n such that we have

$$\{\log(A)\} \leqslant \{n\log(2)\} < \{\log(A+1)\}\,.$$

Let's look at what we have here. The number $X = \log(2)$ is an irrational number[3]. The numbers $a = \{\log(A)\}$ and $b = \{\log(A+1)\}$ satisfy $0 \leqslant a < b < 1$ (notice that $a < b$ because $\log(A) < \log(A+1)$ and by assumption these two numbers have the same floor function). So it suffices that we prove the following fact:

Given an irrational number X, and two numbers a, b satisfying $0 \leqslant a < b \leqslant 1$, there are infinitely many natural numbers n satisfying

$$a \leqslant \{nX\} < b\,.$$

Since X is irrational, you may easily verify that the numbers $\{nX\}$ are distinct for different values of n[4]. Now partition the interval $[0, 1]$ into intervals of some length less than $b - a$. It is pretty intuitive (and it follows from the so-called pigeonhole principle) that there is an interval that contains at least two numbers $\{n_1X\}$ and $\{n_2X\}$, and we may suppose that the former is less than the latter. So we have

$$\{(n_2 - n_1)X\} = \{n_2X\} - \{n_1X\} < b - a\,.$$

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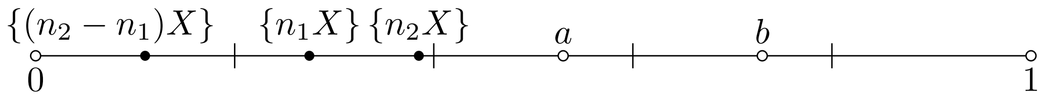
[1]If x is a real number, then we write $\lfloor x \rfloor$ for the *floor function*, which gives the largest integer which is less than or equal to x: for example $\lfloor \pi \rfloor = 3$, $\lfloor 7 \rfloor = 7$, $\lfloor -\pi \rfloor = -4$.

[2]If x is a real number, then we define the *fractional part* $\{x\}$ of x as the difference between x and its floor function. This is a number greater than or equal to 0 and strictly less than 1, for example we have: $\{\pi\} = 0.14\ldots$; $\{7\} = 0$; $\{-\pi\} = 0.85\ldots$.

[3]With the Fundamental Theorem of Arithmetic it is not difficult to prove the following fact: If the decimal logarithm of a natural number is rational, then the number must be a power of 10.

[4]Hint: If $\{nX\} = \{mX\}$ with $n \neq m$, then $X = (\lfloor nX \rfloor - \lfloor mX \rfloor)/(n - m)$.



Then it is not difficult to show that in each of the given intervals there are infinitely many numbers of the form $\{M(n_2 - n_1)X\}$, where $M \geqslant 1$ is an integer[5]. If $n_2 - n_1$ is also positive, then we are done. Else notice that $\{M(n_2 - n_1)X\}$ is non-zero because X is irrational, and hence

$$\{-M(n_2 - n_1)X\} = 1 - \{M(n_2 - n_1)X\}\,.$$

We deduce that each of the given intervals contains also infinitely many numbers of the form $\{-M(n_2 - n_1)X\}$, and we conclude because $-M(n_2 - n_1)$ is positive. This completes the proof!

Finally, some mathematical challenges: Can you generalize the problem addressed in this article by replacing 2 by any integer greater than 1 which is not a power of 10? Can you generalize the problem also to numeral bases other than 10?

Acknowledgements

This article is inspired from the YouTube video "Ogni numero è l'inizio di una potenza di 2" by MATH-segnale (`www.youtube.com/mathsegnale`) and from the article "Elk natuurlijk getal is het begin van een macht van twee" by the author. The images have been created with `wordart.com` and TikZ.

[5]If we subdivide the interval $[0, 1]$ into N intervals of length $\frac{1}{N}$ and if $0 < \ell < \frac{1}{N}$ is irrational, then in any of the intervals there are infinitely many numbers of the form $\{M\ell\}$, where $M \geqslant 1$ is an integer. Recall that these fractional parts are all distinct because ℓ is irrational. By taking the fractional parts of $\ell, 2\ell, 3\ell, \ldots$ we enter each of the intervals (possibly more than once), and for every positive integer a we can start all over again with $t\ell, (t+1)\ell, \ldots$, where $t\ell$ is the smallest multiple of ℓ which is greater than a, for which we must have $0 < \{t\ell\} < \frac{1}{N}$.