The Seven Bridges of Königsberg

In the 18th century the city of Königsberg (now Kaliningrad, Russia) had seven bridges as depicted in Figure 1. The problem which has been called "The Seven Bridges of Königsberg" is the following:

> Is it possible to find a walk through Königsberg that passes exactly once over each of the seven bridges and returns to the starting point?

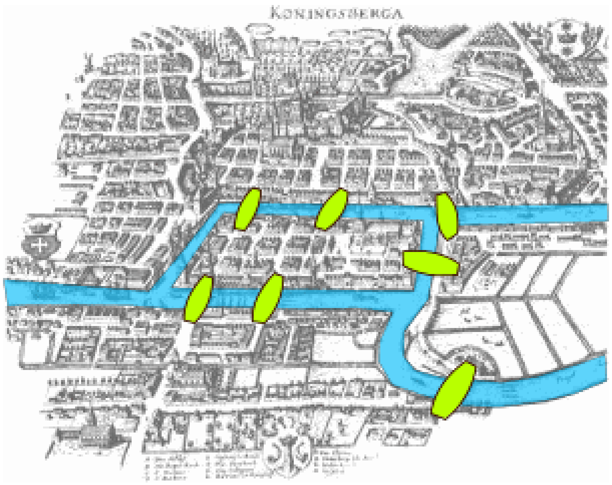

Figure 1: The seven bridges of Königsberg

To better visualize the geography of Königsberg let us draw the corresponding graph (a graph is a mathematical object that is constituted of points — called vertices or nodes — and edges between pairs of these points). Here the points represent the different pieces of land and the edges connecting them are the bridges.

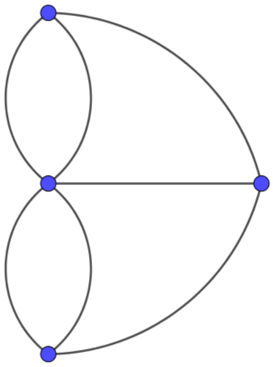

Figure 2: The graph of the seven bridges of Königsberg

The graph version' of the Königsberg bridge problem is the following question:

> Is it possible to traverse the graph of Königsberg in Figure 2 (starting at an arbitrary vertex) using each edge exactly once and return to the starting vertex?

A graph that may be traversed in this way is called a *eulerian graph* and such a circuit is called *eulerian cycle*. The question then amounts to: Is the Königsberg graph a eulerian graph? In other words, does there exist a eulerian cycle?

For a vertex A of the graph, we call the number of edges connected to A the degree of A. Euler proved in 1736 the following result: *If a graph is eulerian, all its vertices have an even degree.* What about the city of Königsberg? The graph of Königsberg has vertices of odd degree, hence it is not a eulerian graph and there is no eulerian cycle! So we cannot find a walk that passes exactly once over each of the seven bridges and returns to the starting point.

The proof of Euler's theorem goes as follows. Let A be a vertex of an eulerian graph, and denote by n_A the number of times that an eulerian cycle passes through A (since we have a cycle we may assume that A is not the starting point). Every time the cycle passes through A two edges are 'consumed', namely one edge to reach A and one edge to leave A. So there are at least $2n_A$ edges at the point A. In other words, the degree of A is at least $2n_A$. Now suppose that the degree of A is strictly larger than $2n_A$. Then there is at least one edge connected to A that is not included in the eulerian cycle, which is impossible. We deduce that the degree of A must be $2n_A$, and in particular it is even. This concludes the proof.

We have seen that having only vertices of even degree is a necessary condition for a graph to be eulerian. Is it also sufficient? In 1873 Hierholzer proved that this is the case for a connected graph (a connected graph is a graph such that for any two points there is a walk on the graph connecting them): *If a connected graph has only vertices of even degree, it is eulerian.* In fact Hierholzer proved that there is a procedure to construct an eulerian cycle. We describe Hierholzer's algorithm on the graph of Figure 3:

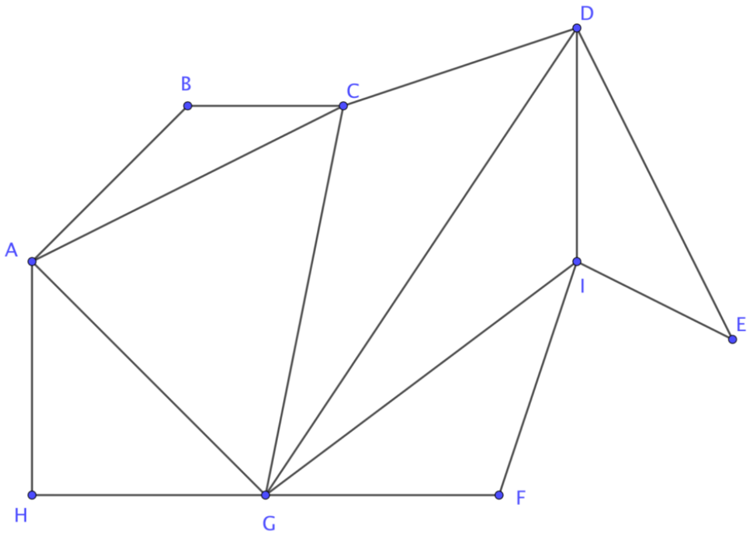

Figure 3: An eulerian graph

We choose a starting vertex (say, the vertex A). We generate a cycle starting at this vertex, such that no edge is used twice (say, the cycle $ABCDEIFGHA$). We highlight the edges contained in this cycle. If all the edges of the graph are highlighted, then we go to the second part of the algorithm. Else, we choose a vertex of the graph to which non-highlighted edges are connected and we generate a similar cycle starting at this vertex, such that no edge is used twice and no highlighted edge is used (say, the cycle $ACGDIGA$). We then highlight the edges of the new cycle. By iterating this procedure we highlight all edges of the graph. What we have done is partitioning the set of edges into cycles. What we are left to do is to merge those cycles in order to build an eulerian cycle. The procedure goes as follows: We start by walking along the first cycle. As soon as we find a point which is on a new cycle, we start walking along the new cycle. In this way we partially complete many cycles (always jumping from a cycle to a cycle we have never used before). If, walking on a cycle, we do not meet new cycles, then we complete the cycle and go back to the preceding cycle. We continue walking on the preceding cycle and go on with the procedure. For example, by merging the cycles $ABCDEIFGHA$ and $ACGDIGA$ for the graph in Figure 3, we find the eulerian cycle $ABCGDIGACDEIFGHA$. Notice that Hierholzer's result has been obtained more than a century after Euler's theorem!

The problem of the seven bridges of Königsberg is an example where graph theory helps answering a concrete question. In fact, there are many applications of graph theory in real life. For example, finding the shortest part on a network can be used for a navigation system, or to select the route for a refuse collection vehicle. Notice that if the network graph it is not eulerian then it can be made eulerian: this is achieved by constructing new edges among pairs of vertices with odd degree (the number of vertices with odd degrees is even, since the sum of all degrees – which counts each edge twice – is even). For the problem of selecting the route of a refuse collection vehicle the new edges correspond to sequences of streets that we have to repeat, and the problem then boils down to adding new edges such that the length of the corresponding streets is the shortest possible (in fact there is also the complication given by one-way streets, and one could also distinguish between streets with few or much traffic…). In any case, graph theory is a powerful tool that does help us solve many real life problems.

Acknowledgements

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Exercises

1. How many bridges must be added (at least) to Königsberg (and where), so that it is possible to find a walk through Königsberg that passes exactly once over each of the bridges and returns to the starting point?

2. Is it possible to find a walk through Königsberg that passes exactly once over each of the seven bridges without returning to the starting point? Equivalently: Is it possible to traverse the graph of Königsberg (starting at an arbitrary vertex) using each edge exactly once without returning to the starting vertex? A graph that may be traversed in this way is called a *semi-eulerian* graph and such a passage is called *eulerian path*.
   (Hint: Adding one suitable edge to a eulerian path gives a eulerian cycle.)

Solutions to the exercises

1. Two bridges suffice. In fact Königsberg's graph is connected, so in order to have a eulerian cycle we only need all vertices to have an even degree. We have four vertices of odd degree, and with two new edges among them we can make those four vertices have even degree. Since we have four vertices of odd degree, adding only one edge would not suffice to make the graph eulerian.

2. Suppose that an eulerian path exists. Adding one edge between the starting and the ending point of the path gives a eulerian cycle. However we have seen in the previous question that adding only one edge cannot make the graph eulerian. We conclude that no eulerian path can exist. In fact, a connected graph is semi-eulerian if and only if exactly two vertices have an odd degree.