MATH AROUND THE CLOCK

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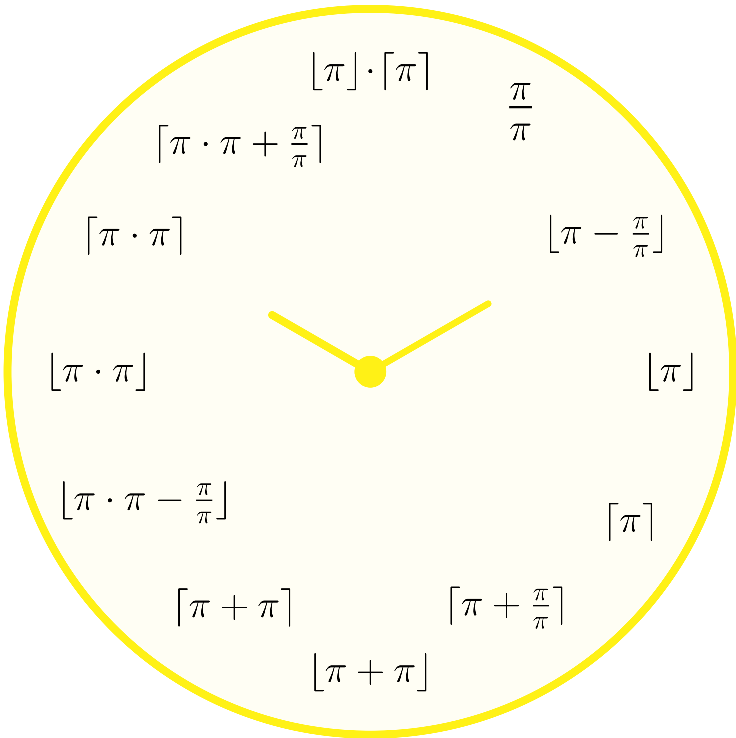

The numbers 1 to 12 written using only π, the basic arithmetic operations and the floor/ceiling functions.

There are infinitely many possibilities for expressing a number! Pupils can invent their personal mathematical clock by writing 1 to 12 in their favourite way...

In the table below we present original mathematical clocks that revolve around a specific theme:

- **Digit-Theme:** One can fix any decimal digit from 1 to 9 and then display all integers 1 to 12 with short expressions involving only that digit and some arithmetic operations. We use the basic arithmetic operations, taking powers, and taking the square-root (notice that with the digits 5,6,7 we only use the basic arithmetic operations). It is also possible to use the digit 0, by applying the factorial identity $0! = 1$.

- **π-Theme:** It is possible to write all integers from 1 to 12 using only π, the basic arithmetic operations and the floor/ceiling functions $\lfloor\,\rfloor$ and $\lceil\,\rceil$.

- **e-Theme:** It is possible to write all integers from 1 to 12 using only Euler's number e, the basic arithmetic operations, taking powers, taking the square-root, and applying the floor/ceiling functions.

- **123-Theme:** It is possible to write all integers from 1 to 12 using only the digits 1,2,3 exactly once and in this order. This involves the basic arithmetic operations, taking powers, taking the square-root, taking the factorial, and applying the floor function. For this theme, we got inspired from [1].

It is also possible to write the integers from 1 *to* 12 *using any given real number.* Indeed, we can always find a short expression for the number 1: for a positive real number which is at most 1, it suffices to take the ceiling function to produce 1; for any real number x greater than 1, we can write 1 as the floor function of $\sqrt[x]{x} = e^{\frac{\log x}{x}}$; negative numbers may be turned positive with the absolute value; for zero we may use its factorial and write $0! = 1$.

Favoring some mathematical expressions over others is a also a matter of personal preference. For example, which of the following expressions

$$2+2 \qquad 2\cdot 2 \qquad 2^2$$

would you choose for the number 4? Of course, one may look for a simple expression or purposely opt for a complicated one: for example, with Euler's identity [3] we can write

$$1 = -e^{\pi i}$$

or, considering the Basel problem [2], we could write

$$6 = \Big(\sum_{n=1}^{\infty} \frac{1}{(\pi n)^2}\Big)^{-1}.$$

Sequences are also a source of inspiration: If n is an integer from 1 to 12, we can write n as $\sqrt{n^2}$ or as $\log_2(2^n)$, for example we can write

$$12 = \sqrt{144} = \log_2(4096)\,.$$

Some mathematical clocks go as far as displaying the first twelve terms of a known sequence (e.g. the Fibonacci sequence), leaving implicit how to make the connection to the integers from 1 to 12: this is analogous to writing *December* to convey the number 12. Another common choice is writing down an equation such that the desired number is the only solution, for example conveying 12 with

$$x^2 + 160 = 24x + 16\,.$$

Notice that some mathematical clocks show equations with more than one solution, but with exactly one solution among the integers from 1 to 12. In general, beware of mathematical inaccuracies, for example 3 is not really $\pi - 0.14$. Nevertheless, everything is allowed to have fun playing with numbers!

Practical tips: There are cheap whiteboard clocks where one can freely write on the clock dial. Alternatively, if one has produced their own mathematical clock, say as a round image, one can use that as clock dial (for example through companies that allow you to pick your own photo).

References

[1] *The Math Clock – 1-2-3 Edition*, Math Clocks & Other Interesting Clocks, `http://www.sbcrafts.net/clocks/`.

[2] P. J. Nahin, *Dr. Euler's Fabulous Formula: Cures Many Mathematical Ills*, Princeton University Press, 2006, 416p.

[3] C. J. Sangwin, *An infinite series of surprises*, Plus Magazine, 1 December 2001, `https://plus.maths.org/content/infinite-series-surprises`.

	1	2	3	4	5	6	7	8	9	10	11	12
1 – Theme	1	$1+1$	$\sqrt{11-1-1}$	$(1+1)^{1+1}$	$\frac{11-1}{1+1}$	$\frac{11+1}{1+1}$	$\sqrt{\frac{111-11}{1+1}-1}$	$\frac{(11+1)(1+1)}{1+1+1}$	$\frac{111-11-1}{11}$	$11-1$	11	$11+1$
2 – Theme	2^{2-2}	2	$2+\frac{2}{2}$	$2\cdot 2$	$2^2+\frac{2}{2}$	2^2+2	$2^2+2+\frac{2}{2}$	$2\cdot 2^2$	$(2+\frac{2}{2})^2$	$2(2^2+\frac{2}{2})$	$\frac{2^{2^2}+2^2+2}{2}$	$2^{2^2}-2^2$
3 – Theme	3^{3-3}	$3-\frac{3}{3}$	3	$\frac{3\cdot 3+3}{3}$	$\frac{3^3+3}{3+3}$	$3+3$	$\frac{3^3+\frac{3}{3}}{3+\frac{3}{3}}$	$\frac{33-\frac{3}{3}}{3+\frac{3}{3}}$	$3\cdot 3$	$\frac{3^3+3}{3}$	$\frac{(3+3)(3+3)-3}{3}$	$3\cdot 3+3$
4 – Theme	4^{4-4}	$\sqrt{4}$	$\frac{4\cdot 4-4}{4}$	4	$4+\frac{4}{4}$	$\frac{44+4}{4+4}$	$\frac{44-4\cdot 4}{4}$	$4+4$	$(4-\frac{4}{4})^{\sqrt{4}}$	$(4+\frac{4}{4})\sqrt{4}$	$\frac{44}{4}$	$4^{\sqrt{4}}-4$
5 – Theme	$\frac{5}{5}$	$\frac{5+5}{5}$	$\frac{5\cdot 5+5}{5+5}$	$\frac{5\cdot 5-5}{5}$	5	$5+\frac{5}{5}$	$\frac{5\cdot 5+5+5}{5}$	$\frac{55+5\cdot 5}{5+5}$	$\frac{5\cdot 5+5\cdot 5-5}{5}$	$5+5$	$\frac{55}{5}$	$\frac{5\cdot 5\cdot 5-5}{5+5}$
6 – Theme	$\frac{6}{6}$	$\frac{6\cdot 6}{6+6+6}$	$\frac{6\cdot 6}{6+6}$	$\frac{6\cdot 6+6\cdot 6}{6+6+6}$	$\frac{6\cdot 6-6}{6}$	6	$6+\frac{6}{6}$	$\frac{66+6\cdot 6-6}{6+6}$	$\frac{66+6\cdot 6+6}{6+6}$	$\frac{66-6}{6}$	$\frac{66}{6}$	$6+6$
7 – Theme	$\frac{7}{7}$	$\frac{77+7}{7\cdot 7-7}$	$\frac{77+7}{77-7\cdot 7}$	$\frac{77-7\cdot 7}{7}$	$\frac{77-7}{7+7}$	$\frac{7\cdot 7-7}{7}$	7	$7+\frac{7}{7}$	$\frac{77+7\cdot 7}{7+7}$	$\frac{77-7}{7}$	$\frac{77}{7}$	$\frac{77+7}{7}$
8 – Theme	$\frac{88-8\cdot 8}{8+8+8}$	$\sqrt{\sqrt{8+8}}$	$\frac{88-8\cdot 8}{8}$	$\sqrt{8+8}$	$\frac{8\cdot 8+8\cdot 8-88}{8}$	$\sqrt{\frac{8\cdot 8+8\cdot 8+8+8}{\sqrt{8+8}}}$	$8-\frac{8}{8}$	8	$8+\frac{8}{8}$	$\sqrt{\frac{888-88}{8}}$	$\frac{88}{8}$	$\frac{8(8\cdot 8+8\cdot 8-8)}{88-8}$
9 – Theme	$\frac{9}{9}$	$\frac{99-9\cdot 9}{9}$	$\sqrt{9}$	$\sqrt{9}+\frac{9}{9}$	$\frac{9\sqrt{9}+\sqrt{9}}{9-\sqrt{9}}$	$9-\sqrt{9}$	$\frac{9\sqrt{9}+\frac{9}{9}}{\sqrt{9}+\frac{9}{9}}$	$\frac{9\cdot 9-\frac{9}{9}}{9+\frac{9}{9}}$	9	$\frac{99-9}{9}$	$\frac{99}{9}$	$9+\sqrt{9}$
π – Theme	$\lfloor\sqrt[\pi]{\pi}\rfloor$	$\lceil\sqrt{\pi}\rceil$	$\lfloor\pi\rfloor$	$\lceil\pi\rceil$	$\lfloor\pi\sqrt{\pi}\rfloor$	$\lfloor\pi+\pi\rfloor$	$\lfloor\pi^{\sqrt{\pi}}\rfloor$	$\lfloor\pi\cdot\pi-\sqrt{\pi}\rfloor$	$\lfloor\pi\cdot\pi\rfloor$	$\lfloor\pi\cdot\pi+\frac{\pi}{\pi\cdot\pi}\rfloor$	$\lfloor\frac{\pi^\pi}{\pi}\rfloor$	$\lfloor\pi\rfloor\cdot\lceil\pi\rceil$
e – Theme	$\lceil e^{-e}\rceil$	$\lceil e-\frac{e}{e}\rceil$	$\lceil e\rceil$	$\lfloor e\sqrt{e}\rfloor$	$\lfloor e+e\rfloor$	$\lceil\frac{e^e}{e}\rceil$	$\lfloor e\cdot e\rfloor$	$\lceil e\cdot e\rceil$	$\lfloor\frac{e^e}{\sqrt{e}}\rfloor$	$\lfloor e\cdot e+e\rfloor$	$\lfloor\frac{e^e}{\sqrt{e}}+e\rfloor$	$\lfloor e^e-e\rfloor$
123 – Theme	$\frac{1+2}{3}$	$1-2+3$	$(-1+2)\cdot 3$	1^2+3	$1\cdot 2+3$	$1+2+3$	$1+2\cdot 3$	$1\cdot 2^3$	$1+2^3$	$\lfloor 12-\sqrt{3}\rfloor$	$-1+2\cdot 3!$	$1!\cdot 2!\cdot 3!$