Paris, nov.3, 2008

Dear Antonella,

Your question about the analytic density of sets of primes can be solved in the following way.

Choose some \(c\) between 0 and 1, e.g. \(c = 1/2\). We shall consider functions of a variable \(x\) with \(x \in I = [0, c]\).

Let \(P\) be the set of all primes. If \(p \in P\), define \(f_p(x) = -1/p^{1+x} \log(x)\).

The \(f_p\)'s have the following properties:

a) they are continuous and \(\geq 0\) on \(I\) and take the value 0 at \(x = 0\).

b) the series \(f(x) = \sum f_p(x)\) converges for every \(x \in I\); the convergence is uniform on every compact subset of \(I\) which does not contain 0.

c) we have \(\lim f(x) = 1\) when \(x \to 0, x \neq 0\). (This shows that \(f\) is discontinuous at 0, since \(f(0) = 0\) because of a).

These properties are enough for constructing a no-density subset of \(P\). More precisely:

Claim - There exists a subset \(Q\) of \(P\) such that, if one defines \(f_Q\) as the sum of the \(f_p\)'s for \(p \in Q\), one has \(\lim \inf f_Q(x) = 0\) and \(\lim \sup f_Q(x) = 1\) when \(x \to 0, x > 0\).

In other words, the upper analytic density of \(Q\) is 1, and its lower analytic density is 0, which is what you wanted (or maybe what you did not want ...).

I feel there should be a functional analysis proof of this claim, à la Banach-Steinhaus (see the comments in Bourbaki EVT V.89, on the method of the "bosse glissante" - indeed, if you draw by computer the graphs of some of the \(f_p\)'s, you shall see they have bumps which are slowly sliding towards 0).

Since I did not manage to find such a nice and clean proof, I have to use a rather pedestrian method. Let me first reformulate the Claim above in a more concrete form:

Claim - There exists \(Q \subset P\) and \(u_n, v_n \in ]0, c]\) with \(u_n, v_n \to 0\), \(f_Q(u_n) < 1/n\) and \(f_Q(v_n) > 1 - 1/n\) for all \(n\).

To prove this, we are going to construct by induction on \(N \geq 1\) a subset \(Q_N\) of \(P\) and points \(u_N, v_N \in ]0, c]\) with the following properties:

i) \(Q_N\) is finite, and contains \(Q_{N-1}\);

ii) \(f_Q(u_n) < 1/n\) and \(f_Q(v_n) > 1 - 1/n\) for every \(n \leq N\).

[Once this is done, we take for \(Q\) the union of the \(Q_N\)'s and we win.]

Let us do the induction step. Note that, if \(M\) is large enough, the sums \(\sum_{p > M} f_p(x_n), n \leq N - 1\), are arbitrary small. Since the conditions "\(< 1/n\)" and "\(> 1 - 1/n\)" define open sets, this implies that there exists \(M_n\) such that \(f_R(u_n) < 1/n\) and \(f_R(v_n) > 1 - 1/n\) for every \(n < N\) and every \(Q\) which is the union of \(Q_{N-1}\) and a set \(Y\) of primes \(> M_n\). We are going to choose \(Q_N\) of that form: \(Q_N = Q_{N-1} \cup Y\) where all the primes in \(Y\) are \(> M_n\). With such a choice, we only have to care about condition ii) for \(u_N\) and \(v_N\). We take \(u_N\) small enough so that \(f_{Q_{N-1}}(u_N) < 1/N\); this is possible since \(f_{Q_{N-1}}(x) \to 0\) when \(x \to 0\).
By replacing $M_n$ by a larger value, if necessary, we shall have $f_{Q_N}(u_N) < 1/N$ for every choice of $Y$, as long as the primes in $Y$ are $> M_n$. We now have to choose $v_N$. This is where we use the fact that $f_p(x) \to 1$ when $x \to 0$. There is a neighborhood $W$ of 0 such that $|1 - f_p(x)| < 1/2N$ for every $x \in W$. If this neighborhood is small enough the sum of the $f_p(x)$, for $p < M_n$, is $< 1/4N$ on $W$. Now, we choose $v_N$ in $W$; we have $|1 - \sum_{p > M_n} f_p(v_N)| < 3/4N$. By choosing for $Y$ a large enough finite set of primes $> M_n$, we have $f_Y(v_N) > 1 - 1/N$ and a fortiori $f_{Q_N} > 1 - 1/N$, where $Q_N = Q_{N-1} \cup Y$.

Voilà. As I said, it is a very pedestrian proof. Clearly there is a more general statement behind this; whenever a convergence is not uniform, one can extract ugly-looking subsequences.

Best wishes

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