Paris, nov.3, 2008

Dear Antonella,

Your question about the analytic density of sets of primes can be solved in the following way.

Choose some $c$ between 0 and 1, e.g. $c = 1/2$. We shall consider functions of a variable $x$ with $x \in I = [0, c]$.

Let $P$ be the set of all primes. If $p \in P$, define $f_p(x) = -1/p^{1+x}.\log(x)$.

The $f_p$'s have the following properties :

a) they are continuous and $\geq 0$ on $I$ and take the value 0 at $x = 0$.

b) the series $f(x) = \sum f_p(x)$ converges for every $x \in I$ ; the convergence is uniform on every compact subset of $I$ which does not contain 0.

c) we have lim $f(x) = 1$ when $x \to 0$, $x \neq 0$. (This shows that $f$ is discontinuous at 0, since $f(0) = 0$ because of a).

These properties are enough for constructing a no-density subset of $P$. More precisely :

*Claim* - There exists a subset $Q$ of $P$ such that, if one defines $f_Q$ as the sum of the $f_p$'s for $p \in Q$, one has lim.inf $f_Q(x) = 0$ and lim.sup $f_Q(x) = 1$ when $x \to 0$, $x > 0$.

In other words, the upper analytic density of $Q$ is 1, and its lower analytic density is 0, which is what you wanted (or maybe what you did not want ...).

I feel there should be a functional analysis proof of this claim, à la Banach-Steinhaus (see the comments in Bourbaki EVT V.89, on the method of the "bosse glissante" - indeed, if you draw by computer the graphs of some of the $f_p$'s, you shall see they have bumps which are slowly sliding towards 0).

Since I did not manage to find such a nice and clean proof, I have to use a rather pedestrian method. Let me first reformulate the Claim above in a more concrete form :

*Claim* - There exists $Q \subset P$ and $u_n, v_n \in ]0, c[$ with $u_n, v_n \to 0$, $f_Q(u_n) < 1/n$ and $f_Q(v_n) > 1 - 1/n$ for all $n$.

To prove this, we are going to construct by induction on $N \geq 1$ a subset $Q_N$ of $P$ and points $u_N, v_N \in ]0, c[$ with the following properties :

i) $Q_N$ is finite, and contains $Q_{N-1}$ ;

ii) $f_{Q_N}(u_n) < 1/n$ and $f_{Q_N}(v_n) > 1 - 1/n$ for every $n \leq N$.

[Once this is done, we take for $Q$ the union of the $Q_N$'s and we win.]

Let us do the induction step. Note that, if $M$ is large enough, the sums $\sum_{p>M} f_p(x_n), n \leq N-1$, are arbitrary small. Since the conditions " $< 1/n$" and " $> 1 - 1/n$" define open sets, this implies that there exists $M_n$ such that $f_R(u_n) < 1/n$ and $f_R(v_n) > 1 - 1/n$ for every $n < N$ and every $Q$ which is the union of $Q_{N-1}$ and a set $Y$ of primes $> M_n$. We are going to choose $Q_N$ of that form : $Q_N = Q_{N-1} \cup Y$ where all the primes in $Y$ are $> M_n$. With such a choice, we only have to care about condition ii) for $u_N$ and $v_N$. We take $u_N$ small enough so that $f_{Q_{N-1}}(u_N) < 1/N$ ; this is possible since $f_{Q_{N-1}}(x) \to 0$ when $x \to 0$.

By replacing $M_n$ by a larger value, if necessary, we shall have $f_{Q_N}(u_N) < 1/N$ for every choice of $Y$, as long as the primes in $Y$ are $> M_n$. We now have to choose $v_N$. This is where we use the fact that $f_P(x) \to 1$ when $x \to 0$. There is a neighborhood $W$ of 0 such that $|1 - f_P(x)| < 1/2N$ for every $x \in W$. If this neighborhood is small enough the sum of the $f_p(x)$, for $p < M_n$, is $< 1/4N$ on $W$. Now, we choose $v_N$ in $W$ ; we have $|1 - \sum_{p \geq M_n} f_p(v_N)| < 3/4N$. By choosing for $Y$ a large enough finite set of primes $> M_n$, we have $f_Y(v_N) > 1 - 1/N$ and a fortiori $f_{Q_N} > 1 - 1/N$, where $Q_N = Q_{N-1} \cup Y$.

Voilà. As I said, it is a very pedestrian proof. Clearly there is a more general statement behind this ; whenever a convergence is not uniform, one can extract ugly-looking subsequences.

Best wishes

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