DISCOVER MULTISETS

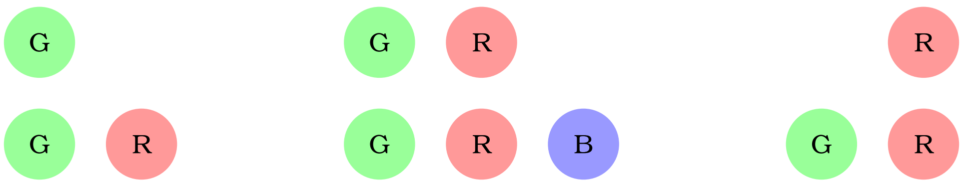
- **Multisets** are like sets, but the elements can be repeated.
- The **multiplicity** of an element is the number of times that it appears in the multiset.
  Note: If an element does not appear, then its multiplicity is 0.

EXAMPLE: If you have two green marbles and one red marble, then you have two distinct elements: 'green' with multiplicity 2, and 'red' with multiplicity 1. In total you have three elements.

EXAMPLE: Think of the coins that are in your wallet: the different elements are the different coin types. Multiplicities matter!

- You have an **inclusion** between two multisets if all elements of the first are also contained in the second with at least the same multiplicity.

EXAMPLE: Consider multisets of colored marbles. The multiset "2 green, 1 red" is included in "2 green, 2 red, 1 blue", but it is not included in "1 green, 2 red".

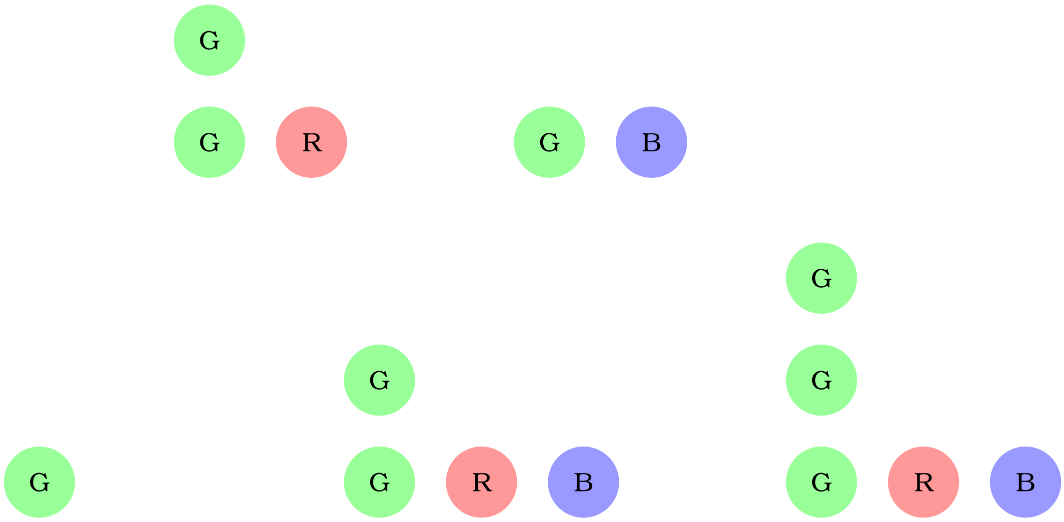


- The **intersection** of two multisets is the largest multiset which is included in both. Simply take the common elements.
- The **union** of two multisets is the smallest multiset which includes both.

- The **sum** of two multisets is the multiset obtained by taking all their elements together.

You can define intersection, union, and sum of several multisets as a straight-forward generalization (and you can also define them for just one multiset, the result being the multiset itself).

EXAMPLE: Consider multisets of colored marbles. The multiset "2 green, 1 red" and the multiset "1 green, 1 blue" have as intersection "1 green", as union "2 green, 1 red, 1 blue", and as sum "3 green, 1 red, 1 blue".



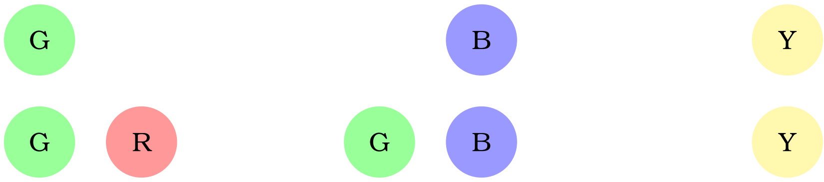
EXERCISE 1: To make some fruity milkshake you can use either 1 kiwi and 1 banana, or 2 kiwis and 3 strawberries. What do you need to buy if you plan to make one of the two milkshakes, but you are still undecided about which one? Which operation of multisets are you doing?

EXERCISE 2: What are the multiplicities for the intersection, the union, and the sum, in terms of those of the original multisets?

- Two or several multisets are **disjoint** if their intersection is empty, which means that they have no common elements.
- Disjoint multisets are **pairwise disjoint** if any two of them are disjoint, so if any two of them have no common elements.

Note: For two disjoint multisets, or for several pairwise disjoint multisets, the union equals the sum.

EXAMPLE: Consider multisets of colored marbles. The multiset "2 green, 1 red", the multiset "1 green, 2 blue", and the multiset "2 yellow" are disjoint, but they are not pairwise disjoint because the first two have one element in common.

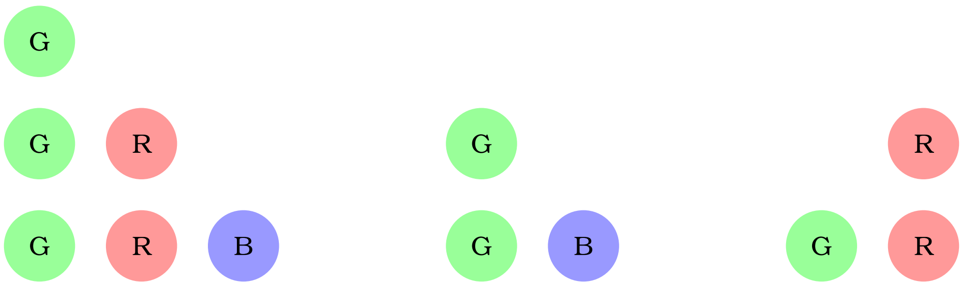


EXERCISE 3: Can you describe the multiplicities of the elements for disjoint multisets and for pairwise disjoint multisets?

EXERCISE 4: If you have three disjoint multisets, does their union equal their sum?

- If a multiset is included in another one, you may consider their **difference**. You simply remove all elements of the former from the latter.

EXAMPLE: Consider multisets of colored marbles. The multiset "3 green, 2 red, 1 blue" minus the multiset "2 green, 1 blue" gives the multiset "1 green, 2 red".



EXERCISE 5: You can also take the difference of any two multisets. How can you do that? A guiding example: If you have some piece of fruits on your shopping list, then consider what happens to the greengrocer's supplies after your purchase.

SOLUTIONS

SOLUTION 1: The total number of elements of a multiset is the sum of the multiplicities of the distinct elements.

SOLUTION 2: You would buy 2 kiwis, 1 banana, and 3 strawberries. You are doing the union of the multisets of the ingredients.

SOLUTION 3: The multiplicity for the intersection is the minimum of the multiplicities. The multiplicity for the union is the maximum of the multiplicities. The multiplicity for the sum is the sum of the multiplicities.

SOLUTION 4: For disjoint multisets, the minimum of the multiplicities of an element is zero because there must be some multiset that does not contain that element (otherwise there would be a common element). For pairwise disjoint multisets, each element can occur in only one of the multisets, otherwise there would be two multisets with some common element. Thus all multiplicities for an element are zero, with at most one exception.

SOLUTION 5: In general the sum may be larger than the union. For two or more multisets, the union equals the sum if and only if the multisets are pairwise disjoint: this condition is necessary and sufficient. Indeed, considering the multiplicities for an element, we want that the maximum of the multiplicities equals the sum of the multiplicities: this happens if and only if the multiplicities with at most one exception are zero. Thus the multisets are pairwise disjoint.

SOLUTION 6: We want to subtract from some multiset M_1 the multiset M_2, and we need to define the difference multiset D. Consider any element: If it occurs in M_1 less often than in M_2, then its multiplicity in D is zero. Otherwise its multiplicity in D is the difference of the multiplicites in M_1 and in M_2. About the given example: If the greengrocer has what you need, you just subtract what you buy. Otherwise you buy as much as possible, leaving the shop without supplies of something. The difference between the multiset "greengrocer's supplies before the purchase" and the multiset "shopping list" is the multiset "greengrocer's supplies after the purchase".