Arithmetic billiards in dimension \( n \)

We assume that the reader is familiar with the setting of the two-dimensional arithmetic billiards.

**Theorem 1.** Let us call \( x_1, x_2, ..., x_n \), the (positive and integer) variables, and let \( a_1, a_2, ..., a_n \), their maximal value. Then the total length of the path in a \( n \)-dimensional cuboid is equal to \( \text{lcm}(a_1, a_2, ..., a_n) \).

**Proof.** First of all, when the path hits one corner, the coordinates of that corner must be:

\[
\begin{cases}
x_1 = 0 \text{ or } x_1 = a_1 \\
x_2 = 0 \text{ or } x_2 = a_2 \\
\vdots \\
x_n = 0 \text{ or } x_n = a_n
\end{cases}
\]

Let us now call \( c_1, c_2, ..., c_n \), the coordinates of the hit corner. Then we know that \( c_1, c_2, ..., c_n \) have to be a positive common multiple of \( a_1, a_2, ..., a_n \), so at least \( \text{lcm}(a_1, a_2, ..., a_n) \). On the other hand, after that amount of unitary steps in the path (independently of the reflections) each coordinate \( x_i \) is a multiple of \( a_i \) and hence we are in a corner.

Let \( I = \{1, \ldots, n\} \) and for every non-empty subset \( J \subseteq I \) write \( \text{lcm}(a_J) \) for the least common multiple of the numbers \( a_i \) with \( i \in J \).

**Theorem 2.** Let us call \( x_1, x_2, ..., x_n \), the (positive and integer) variables, and let \( a_1, a_2, ..., a_n \), their maximal value. The total amount of bouncing points in the \( n \)-dimensional arithmetic billiard is:

\[
\text{lcm}(a_I) \cdot \sum_{J \subseteq I, J \neq \emptyset} (-1)^{|J|+1} \frac{1}{\text{lcm}(a_J)}
\]

**Proof.** The number of bouncing points on the faces \( x_i = 0 \) or \( x_i = a_i \) equals \( \frac{\text{lcm}(a_1, a_2, ..., a_n)}{a_i} \). Similarly, on an edge \( x_i = 0 \) or \( x_i = a_i \) for all \( i \in J \), where \( J \) is a non-empty subset of \( \{1, \ldots, n\} \), equals \( \frac{\text{lcm}(a_J)}{\text{lcm}(a_J)} \). This formula is then obtained by the inclusion-exclusion principle while counting the bouncing points on the faces (we have to keep track of points which are bouncing points for several faces simultaneously).

**Open questions**

For an \( n \)-th dimensional billiard with the path starting at the point \((0, \ldots, 0)\) and each coordinate increasing or decreasing by one at each step:

- In which corner does the ball land?
  This question is probably clarified by considering the highest power of 2 dividing the numbers \( a_i \).

- How many intersection points does the path have?
  Many examples need to be investigate to formulate a conjecture.

In general, what happens if the starting point is not \((0, \ldots, 0)\) but another point in the arithmetic billiard?