ARITHMETIC BILLIARDS

Arithmetic billiards, also known under the name 'Paper Pool', are rectangles where a pointwise ball bounces making a $45°$ angle with the sides. There is a nice interplay between geometry and arithmetic because we take as side lengths two positive integers and because (by placing the origin in a corner and the coordinate axes parallel to the billiard sides) the ball regularly goes through points with integer coordinates. There is no friction and the ball never stops unless it reaches a corner. The path is a zig-zag of segments that usually intersect one another.

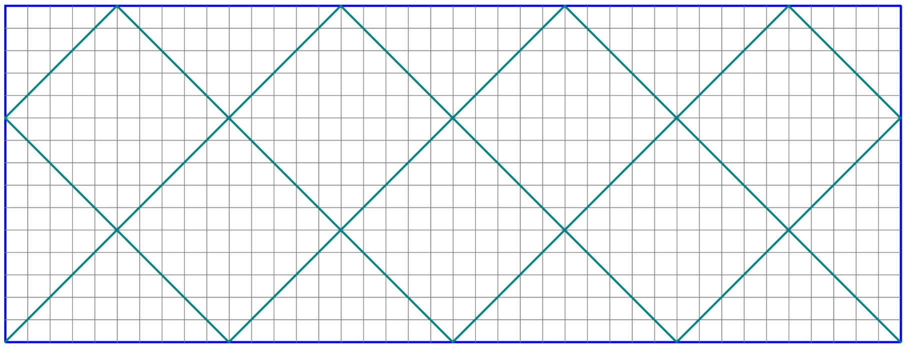

FIGURE 1. Example of path for the arithmetic billiard with sides 40 and 15.

Various authors including Martin Gaardner [1, 6, 7, 5] have studied paths which start at a corner. Such paths (which we call "corner paths") start at a corner and end at a different corner. There are exactly two corner paths. All the other paths (which we call "closed paths") represent periodic trajectories. The shape of a closed path is more complicated with respect to a corner path, and apparently closed paths have not been understood until now.

A nice arithmetical feature of the corner paths is the following. Calling a and b the side lengths of the billiard, the length of the path is $\sqrt{2} \cdot \mathrm{lcm}(a, b)$ and (unless one of the two lengths is a multiple of the other) the smallest distance between the starting point and a self-intersection point of the path equals $\sqrt{2} \cdot \gcd(a, b)$. For closed paths we have analogue results. Namely the length of the path is $2\sqrt{2} \cdot \mathrm{lcm}(a, b)$. Moreover, the path partitions the billiard table into squares, rectangles, and triangles. We usually see squares of two different sizes, whose sum equals $\sqrt{2} \cdot \gcd(a, b)$.

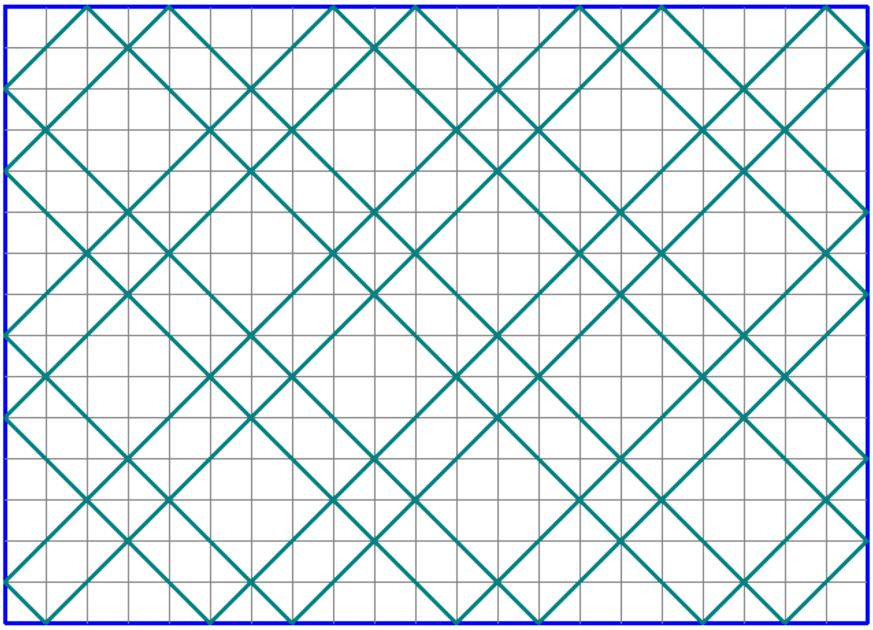

FIGURE 2. Example of closed path for the arithmetic billiard with sides 21 and 15.

The path contains a point of the form $(0, r)$, where r is an integer ranging from 1 to $\gcd(a, b) - 1$, and there are accordingly $\gcd(a, b) - 1$ different closed paths for the same pair (a, b). The sides of the squares are then $\sqrt{2}r$ and $\sqrt{2}(\gcd(a, b) - r)$. Clearly, if $\gcd(a, b)$ is even and r is the half of it, then these two numbers coincide.

Depending on r we are able to determine all the boundary points of a closed path. Moreover, we are able to count the number of triangles, squares, and rectangles of the above partition, and the number of self-intersection points of the path: surprisingly, these quantities only depend on a and b and not on the chosen closed path.

The interested reader may continue this investigation by considering billiard tables of other shapes, or by increasing the dimension: for example we can let a ball bounce inside a parallelepiped. This can also be a mathematical project for talented pupils. In fact, the exploration of arithmetical billiards is already a source of activities for Math Circles [2, 8, 10] and for schools [3, 4].

1. PRELIMINARY REMARKS

1.1. **Setting.** Fix two positive integers a and b, and consider a rectangle (the *billiard*) with sides a and b. We fix coordinates as follows: we place the origin in a billiard corner and let the opposite corner be the point (a, b). We call the billiard sides the a-sides and the b-sides respectively. Let g be the greatest common divisor of a and b.

We consider the trajectory of one point (the *ball*) inside the billiard such that the path consists of segments which make a $45°$ angle with the sides. As a starting point we can take any point in the billiard having integer coordinates. The ball bounces on the billiard sides (making either a left or a right $90°$ turn) and stops only if it lands in a corner. We have (neglecting their orientation) exactly two *corner paths*, where the ball starts in a corner and lands in a different corner (for simplicity we extend the trajectory of a ball landing in one corner to get exactly a corner path). All other paths are the *closed paths*, which correspond to the periodic trajectories.

Notice that we will also consider time for the trajectory, where after a time unit both coordinates change by 1 (either increasing or decreasing).

Any path will have points on the rectangle sides: these are called the *boundary points* of the path. Most paths will have *points of self-intersection*, where two path segments cross perpendicularly.

1.2. **Corner paths.** We now recall the basic properties of the corner paths, see for example [5, 9] for a reference.

The corner paths start in a corner and end in a different corner: there are exactly two corner paths. The length of a corner path is $\sqrt{2}\cdot\mathrm{lcm}(a,b)$, and $\mathrm{lcm}(a,b)$ is is the number of unit squares crossed by the path.

If we start at a given corner we can already predict what the final corner is: the starting corner and the final corner are opposite if and only if the two numbers a/g and b/g are both odd; if a/g is even and b/g is odd, then the starting and the final corner are adjacent to one same a-side; if a/g is odd and b/g is even, then the starting and the final corner are adjacent to one same b-side.

The path is symmetric: if the starting and the ending corner are opposite, then the path is pointsymmetric w.r.t. the center of the rectangle, else it is symmetric with respect to the perpendicular bisector of the side connecting the starting and the ending corner.

There are exactly a/g boundary points (including the corners) on the two a-sides and b/g on the two b-sides. The boundary points are evenly distributed along the rectangle perimeter: the distance along the perimeter (i.e. possibly going around the corner) between two such neighbouring points equals $2g$.

More precisely, the corner path starting in the origin is the intersection of the billiard with the grid of squares having as corners the points (xg, yg), where x, y are integers such that $x + y$ is even (notice that the squares of the grid are oriented at $45°$ w.r.t. the billiard sides). The other corner path is obtained by taking one symmetry of the billiard (namely the one mapping the starting and the final corner to the other two corners).

Unless one number between a and b is a multiple of the other, there are self-intersection points. The smallest distance between the starting point and a self-intersection point of the path then equals $\sqrt{2}\cdot\gcd(a,b)$. Moreover, in this case $\gcd(a,b)$ is the number of unit squares crossed by the first segment of the path up to that point of self-intersection.

2. Boundary points for closed paths

The closed paths do not contain corners, and since there are only finitely many points with integer coordinates these paths correspond to periodic trajectories.

As we will prove later, the path partitions the arithmetic billiards into squares (of up to two distinct sizes) and rectangles, and triangles along the boundary.

• *For a closed path we must have* $g > 1$. Indeed, if $g = 1$ all points in the rectangle with integer coordinates lie on the corner paths.

2.1. **Length of a closed path, number of boundary points.**

Now consider a closed path. We claim that the closed path has boundary points on all sides. Indeed, after a time $2a$ it has been both on the left and on the right b-side, and after a time $2b$ it has been both on the bottom and on the upper a-side. Since a closed path is periodic, we may take the starting point without loss of generality to be on the bottom a-side. Up to changing the orientation of the path we may also suppose that the starting direction is rightwards.

• *The length of a closed path is* $2\sqrt{2} \cdot \mathrm{lcm}(a, b)$. *In particular it depends only on* a, b *and not on the starting point.* Indeed, we are back to the bottom a-side after times which are multiples of $2b$. Moreover since we start the path by going rightwards and we end the path also going rightwards, then we can be back to the starting point only after a multiple of $2a$. So the smallest time at which we are back to the starting point is the least common multiple of $2a$ and $2b$.

• *There are* a/g *boundary points on each* a*-side and* b/g *boundary points on each* b*-side. In particular, these quantities depend only on* a, b *and not on the starting point.* Indeed, starting on the bottom a-side we are back to this side after multiples of $2b$ and considering the length of the path we obtain $\mathrm{lcm}(2a, 2b)/2b = a/g$ bouncing points on the bottom a-side. For the other sides we can make a similar reasoning.

2.2. **Position of the boundary points.**

The numbers a/g and b/g cannot be both even. We distinguish two cases: the case where a/g and b/g are odd, and the case where a/g and b/g are not both odd.

The case where a/g *and* b/g *are odd.* Let r be an integer in the range from 1 to $g - 1$. We claim that the boundary points are as in the following table, where we specify only the coordinate needed to determine the point.

Notice that by varying r we obtain all points of the sides whose side coordinate is not a multiple of g (those other points lie on the two corner paths).

To prove the claim let us place the ball (without loss of generality) on any point of the bottom a-side from our list, and let us prove that the next boundary point of the path (which could be on any of the other three sides) is still in the list. By iterating this procedure, since in the list we have the correct amount of boundary points, we must exhaust all the boundary points.

bottom a-side	$r, 2g-r, \dots, n2g+r, (n+1)2g-r, \dots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \dots, n2g+g-r, n2g+g+r, \dots, \frac{b-g}{2g}2g+g-r$
upper a-side	$g-r, g+r, \dots, n2g+g-r, n2g+g+r, \dots, \frac{a-g}{2g}2g+g-r$
left b-side	$r, 2g-r, \dots, n2g+r, (n+1)2g-r, \dots, \frac{b-g}{2g}2g+r$

FIGURE 3. The boundary points if a/g and b/g are odd. We give the x-coordinate for the a-sides and the y-coordinate for the b-sides. The integer r is in the range from 1 to $g-1$.

Let us then consider the point $(p, 0)$, where p is any integer from 1 to $a-1$ which is not a multiple of g (those other points belong to the corner paths). If we land on the right b-side we have the point $(a, a-p)$, if we land on the upper a-side we have either the point $(p+b, b)$ or the point $(p-b, b)$, if we land on the left b-side we have the point $(0, p)$. It is an easy check to show that if p is in the above list for the bottom a-side, then the other points have their side coordinate in the list.

The case where a/g is odd and b/g is even. If a/g and b/g are not both odd, we suppose without loss of generality that a/g is odd and b/g is even. Let r be an integer in the range from 1 to $g-1$. We claim that the boundary points are as follows, where in the table we specify only the coordinate needed to determine the point:

bottom a-side	$r, 2g-r, \dots, n2g+r, (n+1)2g-r, \dots, \frac{a-g}{2g}2g+r$
right b-side	$g-r, g+r, \dots, (\frac{b}{2g}-1)2g+g-r, (\frac{b}{2g}-1)2g+g+r$
upper a-side	$r, 2g-r, \dots, n2g+r, (n+1)2g-r, \dots, \frac{a-g}{2g}2g+r$
left b-side	$r, 2g-r, \dots, (\frac{b}{2g}-1)2g+r, (\frac{b}{2g}-1)2g+2g-r$

FIGURE 4. The boundary points if a/g is odd and b/g is even. We give the x-coordinate for the a-sides and the y-coordinate for the b-sides. The integer r is in the range from 1 to $g-1$.

The proof is analogous to the previous case, therefore we leave it as an exercise.

3. SHAPE OF A CLOSED PATH

3.1. **The grid structure.** Since we know the boundary points, we know exactly the segments which form the closed path. They are in particular diagonal lines with slope 1 and -1. The distance between parallel lines is alternatively $\sqrt{2}r$ and $\sqrt{2}(g-r)$. Moreover, the point $(r, 0)$ is on the path. Then the path segments form a grid which partitions the billiard table into (diagonally oriented) squares of sides $\sqrt{2}r$ and $\sqrt{2}(g-r)$ respectively, (diagonally oriented) rectangles of sides $\sqrt{2}r$ and $\sqrt{2}(g-r)$, triangles around the border which are half of one of the two types of squares, and triangles at the corners of the billiard table which are a quarter of one of the two types of squares. We call *corner triangles* the triangles containing the corners, and the further triangles along the boundary *side triangles*.

3.2. **Triangles along the boundary.** The boundary points are evenly distributed along the rectangle perimeter (i.e. possibly going around the corner): the distance between any two of them is alternatively $2r$ and $2g - 2r$.

Corner triangles. Suppose that a/g and b/g are odd. Notice that the formula for the boundary points implies that two of the corner triangles (which are half of a square) have legs r and the other two have legs $g - r$. What is clear from the distribution of the boundary points is the following: opposite triangle corners are equal if a/g and b/g are both odd; if a/g is even and b/g is odd, then the triangle corners adjacent to one same a-side are equal; if a/g is odd and b/g is even, then the triangle corners adjacent to one same b-side are equal.

Side triangles. Now consider the triangles along the boundary which are not corner triangles. Such triangles are half of a square and the length of their hypothenuse is $2r$ and $2g - 2r$ respectively.

If $r = g/2$, then the side triangles have all hypothenuse g: we get $a/g - 1$ triangles along each a-side and $b/g - 1$ triangles along each b-side. On the other hand, if $r \neq g/2$, then there are $a/g - 1$ side triangles of each type along the a-sides and there are $b/g - 1$ side triangles of each type along the b-sides.

3.3. **The number of squares and rectangles.** If $r = g/2$, then in the partition all rectangles are squares and all squares have the same side length $g/\sqrt{2}$. Else, we have in the partition rectangles which are not squares (with sides $\sqrt{2}r$ and $\sqrt{2}(g - r)$) and we have squares of two sizes (with side lengths $\sqrt{2}r$ and $\sqrt{2}(g - r)$).

Suppose that $r \neq g/2$. To count the squares we go along the direction of a billiard side that has two different triangle corners. Call S this side, and S' the other side. We consider a 'stripe of squares', by which we mean a sequence of squares whose centers have the same S'-coordinate. In such a stripe smaller and larger squares alternate. We know that S is an odd multiple of g and hence the number of squares that we have in the stripe is $(S - g)/g$, and we have the same amount of the two types of squares. We are left to count how many stripes of squares we have.

If the other side S' is an odd multiple of g we have $2(S' - g)/2g = (S' - g)/g$ stripes of squares. If the other side S' is an even multiple of g, we have $2(S'/2g) - 1 = (S' - g)/g$ stripes of squares.

The total number of squares is then

$$\frac{(a-g)(b-g)}{g^2},$$

and we have the same amount of squares of the two types. We leave it to the reader as an exercise to count the non-square rectangles.

If $r = g/2$, then the rectangles are also squares, and all squares have the same size. A similar computation shows that the total number of squares is

$$\frac{2ab}{g^2} - \frac{a+b}{g} + 1\,.$$

3.4. **The number of self-intersection points.** If $a \neq b$, then there are self-intersection points of the path. It is easier to count them together with the boundary points, and then subtract the $2(a+b)/g$ boundary points. Notice that all these points are exactly the vertices of (possibly more than one) rectangle. Consider 'rectangle stripes' in the direction of the a-side (namely, the sequences of rectangles whose centers have the same b-coordinate): they contain a/g rectangles and there are b/g such stripes. The rectangles in each stripe have $3a/g + 1$ distinct vertices, and any two rectangle stripes have a/g common vertices. We deduce that the total number of distinct rectangle vertices is $2ab/g^2+(a+b)/g$ and hence the number of self-intersection points is $2ab/g^2-(a+b)/g$.

3.5. **The number of closed paths.** We have given parametric formulas for the boundary points, where the parameter r is in the range from 1 to $g-1$. We can easily deduce the following:
• *There are $g-1$ closed paths.*
Recall that the length of the path, the number of boundary points, the number of self-intersection points, the number of squares and so on do not depend on the starting point.

Finally notice that if $g = 2$, then there is only one closed path, which consists of the grid of squares among the points (x, y) in the billiards such that $x + y$ is odd.

Exercises for the reader

(1) What is the shape of a closed path in an arithmetic billiard which is a square?
(2) Consider a closed path in an arithmetic billiard with integer sides a, b. Count, in the partition provided by the path, the number of rectangles which are non-squares.
(3) Consider the three-dimensional generalization of the arithmetic billiard, where the ball bounces in a parallelepiped with integer sides a, b, c, moving in a direction making a $45°$ angle with the sides. Show that a path starting in a corner ends in a corner, and that the length of the path is $\sqrt{3} \cdot \operatorname{lcm}(a, b, c)$.

Summary. Arithmetic billiards are rectangles where a pointwise ball moves and bounces making a $45°$ angle with the sides. We consider rectangles of integer sides a and b and we require that the ball goes through points with integer coordinates (by taking one rectangle corner as the origin and the opposite corner as the point (a, b)). The geometry of the path is very interesting. In particular, the length of the path is related to the $\operatorname{lcm}(a, b)$ and there are segments in the path which give the $\gcd(a, b)$. The novelty of this article with respect to the literature is that we consider all possible paths, and not only those ending in a corner.

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SOLUTIONS TO THE EXERCISES FOR THE READER

(1) We fix coordinates as follows: we place the origin in a square corner and let the opposite corner be the point (a, a). Then the path is the rectangle with corners $(r, 0)$, $(a, a - r)$, $(a - r, a)$, $(0, r)$, where r is an integer in the range from 1 to $a - 1$.

(2) If the path goes through the point $(0, g/2)$, then we have seen that all rectangles in the partition are squares and the requested number is 0. Now suppose that $r \neq g/2$. We consider the 'rectangle stripes' in the direction of the a-side (namely, the sequences of rectangles whose centers have the same b-coordinate): each stripe contains a/g rectangles. We have b/g such stripes so we deduce that the number of rectangles is ab/g^2.

(3) Consider coordinates such that the origin is in one parallelepiped corner and the opposite corner is the point (a, b, c). Without loss of generality, start in the origin. Consider that in a time unit each coordinate is either increased or decreased by 1. Then after any multiple of $\operatorname{lcm}(a, b, c)$ time units we are in a corner because the coordinates of the corners are:

$$\begin{cases} x = 0 \;\; \mathit{or} \;\; x = a \\ y = 0 \;\; \mathit{or} \;\; y = b \\ z = 0 \;\; \mathit{or} \;\; z = c\,. \end{cases}$$

On the other hand, considering these coordinates, we are in a corner only after a common multiple of a, b, and c, so the first occurrence is after $\operatorname{lcm}(a, b, c)$ time units. The total length of the path is then $\sqrt{3} \cdot \operatorname{lcm}(a, b, c)$. Notice that we can determine which one is the final corner by looking at the parity of the numbers $\operatorname{lcm}(a, b, c)/a$, $\operatorname{lcm}(a, b, c)/b$, and $\operatorname{lcm}(a, b, c)/c$.