# 5-CON TRIANGLES

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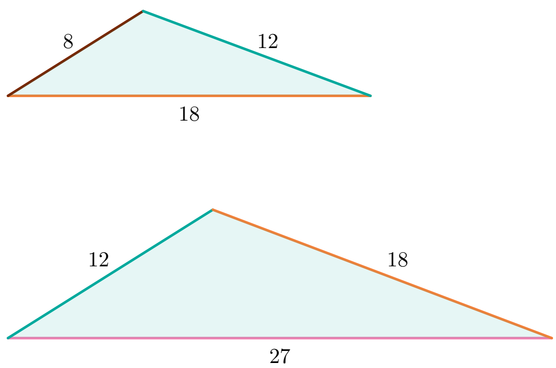
*Specifying three angles and two sides does not determine a triangle up to congruence, but only up to similarity. There are problematic examples among acute/right/obtuse triangles and among triangles with integer sides.*

Consider two triangles such that two pairs of sides are equal in length, and all angles are equal in measurement. The triangles must be similar because they have the same angles but they need not to be congruent, as the following example shows:




Take a triangle with sides

$$(1, \lambda, \lambda^2)$$

for some positive real number $\lambda$. Rescaling it by $\lambda$ we obtain the sides

$$(\lambda, \lambda^2, \lambda^3)$$

thus all angles and two sides are the same. The two triangles are similar but they are not congruent as soon as $\lambda \neq 1$. Up to switching the triangles, we may suppose $\lambda > 1$.
We have to choose $\lambda$ carefully. Indeed, a triple $(a, b, c)$ of positive real numbers such that $a \leqslant b \leqslant c$ consists of the sides of a triangle if and only if $c < a + b$ holds. In our situation (since we are assuming $\lambda > 1$) we must have $\lambda^2 < 1 + \lambda$ and hence

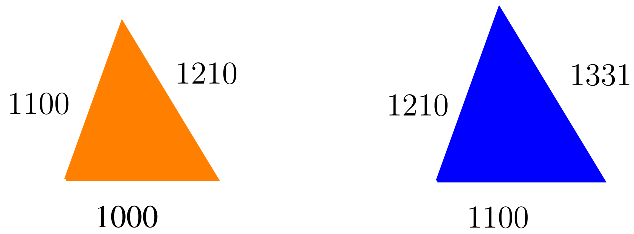
$$1 < \lambda < \frac{1+\sqrt{5}}{2} \sim 1.6\,.$$

All examples are of this form: the sides must be respectively

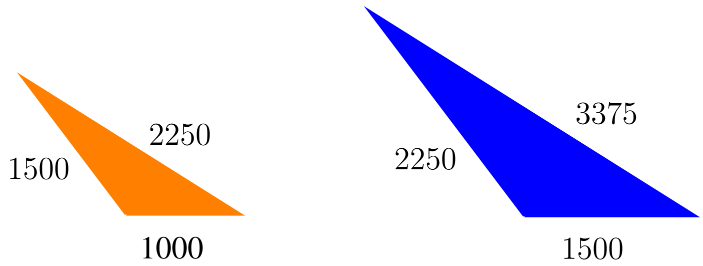
$$(r, r\lambda, r\lambda^2) \qquad \text{and} \qquad (r\lambda, r\lambda^2, r\lambda^3)$$

for some positive real numbers $r$ and $\lambda$. We must require $\lambda \neq 1$ because we do not want congruent triangles. It is crucial that two side ratios equal the similarity factor $\lambda$ because after rescaling we need to obtain two sides again.
Thus we must have two similar triangles that are not isosceles and such that the geometric mean of the greatest and the smallest side equals the third side (and the similarity factor must be the square root of the ratio between the greatest and the smallest side).

• By setting $r = 1000$ and $\lambda = 1.1$, we obtain two acute triangles with integer sides:




• By setting $r = 1000$ and $\lambda = 1.5$, we obtain two obtuse triangles with integer sides:



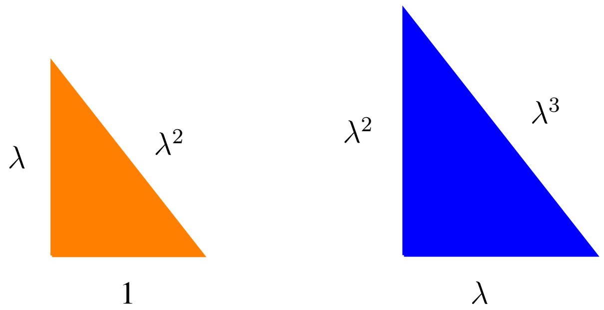

• A triple $(a, b, c)$ of positive real numbers such that $a \leqslant b \leqslant c$ consists of the sides of a right triangle if and only if $a^2 + b^2 = c^2$ holds. Thus we can find a right triangle with sides $(1, \lambda, \lambda^2)$ and hypothenuse $\lambda^2$ because there is a positive real number $\lambda$ satisfying

$$1^2 + \lambda^2 = \lambda^4 \,.$$

We first solve the quadratic equation $1 + x = x^2$, which gives $x = \frac{1 \pm \sqrt{5}}{2}$. Setting the positive solution equal to $\lambda^2$, we get

$$\lambda = \sqrt{\frac{1+\sqrt{5}}{2}} \sim 1.3 \,.$$

With this choice of $\lambda$ and by setting $r = 1$ we obtain an example consisting of two right triangles:




Notice that the number $\frac{1+\sqrt{5}}{2}$ is the golden ratio! As a historical note, right 5-Con triangles are also called *Kepler triangles*.

## Questions for the reader

(1) Show that there are infinitely many pairs of 5-Con triangles, even up to scaling.
(2) Show that there are no examples of right 5-Con triangles with integral sides.
(3) Show that there are no 5-Con triangles that are equilateral, or isosceles.
(4) Show that the arithmetic mean, the geometric mean and the harmonic mean of two positive real numbers are the lengths of the sides of a right triangle if and only if that triangle is a 5-Con triangle.

## References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[4] Thebault, Victor; Pinzka, C. F. (1955). *E1162*. The American Mathematical Monthly. Mathematical Association of America. 62, Dec. 1955 (10): 729–730. Retrieved 6 March 2017.